

Quasi-simple finite groups of essential dimension 3

Yuri Prokhorov *

Steklov Mathematical Institute, Russia

Moscow State Lomonosov University, Russia

National Research University Higher School of Economics, Russia

e-mail: `prokhorov@mi.ras.ru`

To the memory of Professor Alfred Lvovich Shmel'kin

UDK 512.76

Keywords: essential dimension, group, algebraic variety, representation, Cremona group

Abstract

We classify quasi-simple finite groups of essential dimension 3.

1 Introduction

This paper is based on the author's talk given at the Magadan conference.

Let G be a finite group and let V be a faithful representation of G regarded as an algebraic variety. A *compression* is a G -equivariant dominant rational map $V \dashrightarrow X$ of faithful G -varieties. The *essential dimension* of G , denoted $\text{ed}(G)$, is the minimal dimension of all faithful G -varieties X appearing in compressions $V \dashrightarrow X$. This notion was introduced by J. Buhler and Z. Reichstein [BR97] in relation to some classical problems in the theory of polynomials. It turns out that the essential dimension depends only on the group G , i.e. it does not depend on the choice of linear representation V [BR97, Theorem 3.1].

*The author was partially supported by RFBR 15-01-02164 and 15-01-02158 and by the Russian Academic Excellence Project “5-100”.

The computation of the essential dimension is a challenging problem of algebra and algebraic geometry. The finite groups of essential dimension ≤ 2 have been classified (see [Dun13]). Simple finite groups of essential dimension 3 have been recently determined by A. Beauville [Bea14] (see also [Ser10], [Dun10]):

1.1 Theorem. *The simple groups of essential dimension 3 are \mathfrak{A}_6 and possibly $\mathrm{PSL}_2(11)$.*

The essential dimension of p -groups was computed by Karpenko and Merkurjev [KM08]. In this short note we find all the finite quasi-simple groups of essential dimension 3.

1.2 Definition. A group G is said to be *quasi-simple* if G is perfect, that is, it equals its commutator subgroup, and the quotient of G by its center is a simple non-abelian group.

The main result of this paper is the following.

1.3 Theorem. *Let G be a finite quasi-simple non-simple group. If $\mathrm{ed}(G) = 2$, then $G \simeq 2.\mathfrak{A}_5$. If $\mathrm{ed}(G) = 3$, then $G \simeq 3.\mathfrak{A}_6$.*

1.4 Notation. Throughout this paper the ground field is supposed to be the field of complex numbers \mathbb{C} . We employ the following standard notations used in the group theory.

- μ_n denotes the multiplicative group of order n (in \mathbb{C}^*),
- \mathfrak{A}_n denotes the alternating group of degree n ,
- $\mathrm{SL}_n(q)$ (resp. $\mathrm{PSL}_n(q)$) denotes the special linear group (resp. projective special linear group) over the finite field \mathbf{F}_q ,
- $n.G$ denotes a non-split central extension of G by μ_n ,
- $z(G)$ (resp. $[G, G]$) denotes the center (resp. the commutator subgroup) of a group G .

All simple groups are supposed to be non-cyclic.

2 Proof of Theorem 1.3

The following assertion is an immediate consequence of the corresponding fact for simple groups [Pro12].

2.1 Proposition. *Let X be a three-dimensional rationally connected variety and let $G \subset \text{Bir}(X)$ be a finite quasi-simple non-simple group. Then G is isomorphic to one of the following:*

$$\text{SL}_2(7), \text{SL}_2(11), \text{Sp}_4(3), 2.\mathfrak{A}_5, n.\mathfrak{A}_6, n.\mathfrak{A}_7 \text{ with } n = 2, 3, 6. \quad (2.1.1)$$

Proof. We may assume that G biregularly (and faithfully) acts on X . Let $Y := X/z(G)$ and $G_1 := G/z(G)$. Then Y is a three-dimensional rationally connected variety acted on by a finite simple group G_1 . Then according to [Pro12] G_1 belongs to the following list:

$$\text{PSL}_2(7), \text{PSL}_2(8), \text{PSL}_2(11), \text{PSp}_4(3), \mathfrak{A}_5, \mathfrak{A}_6, \mathfrak{A}_7. \quad (2.1.2)$$

Since G_1 is perfect, there exists the *universal covering group* \tilde{G}_1 , that is, a central extension of G_1 such that for any other extension \hat{G}_1 there is a unique homomorphism $\tilde{G}_1 \rightarrow \hat{G}_1$ of central extensions (see e.g. [Kar93, §11.7, Theorem 7.4]). The kernel of $\tilde{G}_1 \rightarrow G_1$ is the Schur multiplier $M(G_1) = H^2(G_1, \mathbb{C}^*)$ of G_1 . Thus G is uniquely (up to isomorphism) determined by $G/z(G)$ and the homomorphism $M(G_1) \rightarrow z(G)$. It is known that $M(G_1) \simeq \mu_2$ in all the cases (2.1.2) except for \mathfrak{A}_6 and \mathfrak{A}_7 where the Schur multiplier is isomorphic to μ_6 , and $\text{PSL}_2(8)$ where the Schur multiplier is trivial (see [Kar93, §12.3, Theorem 3.2, §16.3, Theorem 3.2] and [CCN⁺85]). This gives us the list (2.1.1). \square

2.1.3 Remark. We do not assert that all the possibilities (2.1.1) occur. Using the technique developed in the works [Pro12], [PS16c], [PS16b], [PS16a] it should be possible to obtain a complete classification of actions of quasi-simple groups on rationally connected threefolds. However, this is much more difficult problem.

2.1.4 Corollary. *Let G be a group satisfying the conditions of 2.1. Then G has an irreducible faithful representation. The minimal dimension of such a representation V is given by the following table [CCN⁺85]:*

G	$2.\mathfrak{A}_5$	$3.\mathfrak{A}_6$	$\text{SL}_2(7), \text{Sp}_4(3), 2.\mathfrak{A}_6, 2.\mathfrak{A}_7$	$\text{SL}_2(11), 6.\mathfrak{A}_6, 3.\mathfrak{A}_7, 6.\mathfrak{A}_7$
$\dim V$	2	3	4	6

2.2 Construction. Let G be a finite quasi-simple non-simple group having a faithful irreducible representation V . Let $\psi : V \dashrightarrow X$ be a compression with $\dim X = \text{ed}(G)$. Applying an equivariant resolution of singularities (see [AW97]) we may assume that X is smooth (and projective). Furthermore, consider the compactification $\mathbb{P} := \mathbb{P}(V \oplus \mathbb{C})$ and let $X \xleftarrow{\varphi} Y \xrightarrow{f} \mathbb{P} \supset V$ be an equivariant resolution of ψ , where f is a birational morphism and Y is smooth and projective. We also may assume that f is passed through the blowup $\tilde{f} : \tilde{\mathbb{P}} \rightarrow \mathbb{P}$ of $0 \in V \subset \mathbb{P}$. Let $\tilde{E} \subset \tilde{V}$ be the \tilde{f} -exceptional divisor and let $E \subset Y$ be its proper transform, and let $B := \varphi(E)$. Thus we have the following G -equivariant diagram:

$$\begin{array}{ccccccc}
 & & E & \subset & Y & \longrightarrow & \tilde{\mathbb{P}} \supset \tilde{E} \\
 & \swarrow & & & \searrow & & \downarrow \tilde{f} \\
 B & \subset & X & \xleftarrow{\varphi} & Y & \xrightarrow{f} & \mathbb{P} \supset V \ni 0
 \end{array}$$

(Note: A curved arrow also connects E to \tilde{E} .)

The action of $z(G)$ on $\tilde{E} \simeq \mathbb{P}(V)$ and on E is trivial because V is an irreducible representation. Hence, $G/z(G)$ faithfully acts on E . By assumption the action of $z(G)$ on X is faithful. Hence $B \neq X$ and so B is a rationally connected variety of dimension $< \text{ed}(G)$.

2.3 Proposition. *Let G be a finite quasi-simple non-simple group having a faithful irreducible representation. Then $G/z(G)$ acts faithfully on a rationally connected variety of dimension $< \text{ed}(G)$.*

Proof. Let V be a faithful irreducible representation of G and let $\psi : V \dashrightarrow X$ be a compression with $\dim X = \text{ed}(G)$. Apply the construction 2.2. Assume that G has a fixed point $P \in X$. Then G has faithful representation on the tangent space $T_{P,X}$. Let $T_{P,X} = \oplus T_i$ be the decomposition in irreducible components. At least one of them, say T_1 is non-trivial. Then $G/z(G)$ faithfully acts on $\mathbb{P}(T_1)$, where $\mathbb{P}(T_1) < \dim X = \text{ed}(G)$. Thus we may assume that G has no fixed points on X . By the construction 2.2 the variety B is rationally connected and $\dim B < \text{ed}(G)$. Since G has no fixed points on B and the group $G/z(G)$ is simple, its action on B must be effective. \square

Comparing the list (2.1.1) with Theorem 3.1 we obtain the following.

2.4 Corollary. *Let G be a finite quasi-simple non-simple group with $\text{ed}(G) \leq 3$. Then for G we have one of the following possibilities:*

$$\text{SL}_2(7), \quad n.\mathfrak{A}_6 \text{ with } n = 2, 3, 6. \quad (2.4.1)$$

Now we consider the possibilities of (2.4.1) case by case.

2.5 Lemma. $\text{ed}(2.\mathfrak{A}_5) = 2$ and $\text{ed}(3.\mathfrak{A}_6) = 3$.

Proof. Let us prove, for example, the second equality. Since $3.\mathfrak{A}_6$ has a faithful three-dimensional representation, $\text{ed}(3.\mathfrak{A}_6) \leq 3$. On the other hand, \mathfrak{A}_6 cannot effectively act on a rational curve. Hence, by Proposition 2.3 $\text{ed}(3.\mathfrak{A}_6) \geq 3$. \square

2.5.1 Lemma. *Let G be a quasi-simple non-simple group. Assume that $G \not\cong 2.\mathfrak{A}_5, 3.\mathfrak{A}_6$. Assume also that G contains a subgroup \bar{H} such that*

- (i) \bar{H} is not abelian but its image $H \subset G/\mathbf{z}(G)$ is abelian,
- (ii) for any action of $G/\mathbf{z}(G)$ on a rational projective surface the subgroup $H \subset G/\mathbf{z}(G)$ has a fixed point.

Then $\text{ed}(G) \geq 4$.

Proof. Since $\bar{H}/(\mathbf{z}(G) \cap \bar{H}) = H$, we have $\mathbf{z}(G) \cap \bar{H} \supset [\bar{H}, \bar{H}]$ and $[\bar{H}, \bar{H}] \neq \{1\}$ (because \bar{H} is not abelian). Assume that $\text{ed}(G) = 3$. Apply construction 2.2. From the list (2.1.1) one can see that $G/\mathbf{z}(G)$ cannot faithfully act on a rational curve and by Corollary 2.1.4 G has no fixed points on X . Hence, B is a (rational) surface. By (ii) the group \bar{H} has a fixed point, say P , on $B \subset X$. There is an invariant decomposition $T_{P,X} = T_{P,B} \oplus T_1$, where $\dim T_1 = 1$. The action of $[\bar{H}, \bar{H}]$ on $T_{P,B}$ and T_1 is trivial. Hence it is trivial on $T_{P,X}$ and X , a contradiction. \square

2.6 Proposition. $\text{ed}(\text{SL}_2(7)) = 4$.

2.6.1 Lemma. *Let S be a smooth projective rational surface admitting the action of $\text{PSL}_2(7)$. Let $H \subset \text{PSL}_2(7)$ be a subgroup isomorphic to $\mu_2 \times \mu_2$. Then H has a fixed point on S .*

Proof. Since H is abelian, according to [KS00] it is sufficient to show the existence of a fixed point on some birational model of S . By Theorem 3.1 we may assume that S is either \mathbb{P}^2 or some special del Pezzo surface of degree 2 (see 3.1(iii)). In the former case, $\mathbb{P}^2 = \mathbb{P}(W)$, where W is a three-dimensional irreducible representation of $\text{PSL}_2(7)$. Then the abelian group $H \simeq \mu_2 \times \mu_2$ has a fixed point on $\mathbb{P}^2 = \mathbb{P}(W)$. Thus we assume that S is a del Pezzo surface of degree 2.

Let $\alpha \in H$ be an element of order 2. First assume that α has a curve C of fixed points. The image $\pi(C)$ under the anticanonical double cover $\pi : S \rightarrow \mathbb{P}^2$ must be a line (because the action on \mathbb{P}^2 is linear). Let $\alpha' \in H$, $\alpha' \neq \alpha$ be another element of order 2. Then $\alpha'(C)$ is also a curve C of

α -fixed points and $\pi(\alpha'(C))$ is also a line. Hence, $\pi(\alpha'(C)) = \pi(C)$ and $\pi^{-1}(\pi(C))$ contains $\alpha'(C)$ and C . Since $\pi^{-1}(\pi(C)) \sim -K_S$ and the fixed point locus of α is smooth, we have $\alpha'(C) = C = \pi^{-1}(\pi(C)) \sim -K_S$ and it is an ample divisor. Note that all the elements of order 2 are conjugate in $\mathrm{PSL}_2(7)$. Hence α' also has a curve of fixed points, say C' , and $C' \sim -K_S$. Then the intersection points $C \cap C'$ are fixed by $H = \langle \alpha, \alpha' \rangle$.

Thus we may assume that any element $\alpha \in H$ of order 2 has only isolated fixed points. The holomorphic Lefschetz fixed point formula shows that the number of these fixed points equals $4\chi(\mathcal{O}_S) = 4$. Then by the topological Lefschetz fixed point formula

$$\mathrm{Tr}_{H^2(S, \mathbb{C})} \alpha^* = 2.$$

Since $\dim H^2(S, \mathbb{C}) = 8$ and all the eigenvalues of α^* equal ± 1 , its determinant must be equal to -1 and so we have a non-trivial character of the group $\mathrm{PSL}_2(7)$. This contradicts the fact that $\mathrm{PSL}_2(7)$ is simple. \square

Proof of Proposition 2.6. Consider the subgroup $\bar{H} \subset \mathrm{SL}_2(7)$ generated by the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 5 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 5 \\ 1 & -1 \end{pmatrix}$$

Let H be its image in $\mathrm{PSL}_2(7)$. It is easy to check that $A^2 = B^2 = -I$ and $[A, B] = -I$. Hence \bar{H} is isomorphic to the quaternion group Q_8 and $H \simeq Q_8/\mathrm{z}(Q_8) \simeq \mu_2 \times \mu_2$. By Lemma 2.6.1 the group \bar{H} has a fixed point on $B \subset X$. Hence we can apply Lemma 2.5.1. \square

2.7 Proposition. $\mathrm{ed}(2.\mathfrak{A}_6) = 4$, $\mathrm{ed}(6.\mathfrak{A}_6) \geq 4$.

Proof. We are going to apply Lemma 2.5.1. Let S be a projective rational surface acted by $G/\mathrm{z}(G) = \mathfrak{A}_6$. By Theorem 3.1 we may assume that $S \simeq \mathbb{P}^2$. The group $2.\mathfrak{A}_6$ is isomorphic to $\mathrm{SL}_2(9)$ (see [CCN⁺85]). As in the proof of Proposition 2.6 take the subgroup $\bar{H} \subset \mathrm{SL}_2(9)$ generated by the matrices

$$A = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 4 & 1 \\ 1 & -4 \end{pmatrix} \quad (2.7.1)$$

and apply Lemma 2.5.1.

In the case $G = 6.\mathfrak{A}_6$, let $Z \subset \mathrm{z}(G)$ be the subgroup of order 3. Then $6.\mathfrak{A}_6/Z \simeq 2.\mathfrak{A}_6 \simeq \mathrm{SL}_2(9)$. Let \hat{H} be the inverse image of the subgroup of $\mathrm{SL}_2(9)$ generated by A and B from (2.7.1) and let $\bar{H} \subset \hat{H}$ be the Sylow 2-subgroup. Then, as above, $\bar{H} \simeq Q_8$ and we can apply Lemma 2.5.1. \square

2.7.2 Remark. Since $6.\mathfrak{A}_6$ has a six-dimensional faithful representation, one has $\mathrm{ed}(6.\mathfrak{A}_6) \leq 6$. However, we are not able to compute the precise value.

3 Appendix: simple subgroups in the plane Cremona group

The following theorem can be easily extracted from the classification [DI09]. For convenience of the reader we provide a relatively short and self-contained proof.

3.1 Theorem ([DI09]). *Let $G \subset \text{Cr}_2(\mathbb{C})$ be a finite simple subgroup. Then the embedding $G \subset \text{Cr}_2(\mathbb{C})$ is induced by one of the following actions:*

- (i) \mathfrak{A}_5 , $\text{PSL}_2(7)$, or \mathfrak{A}_6 acting on \mathbb{P}^2 ,
- (ii) \mathfrak{A}_5 acting on the del Pezzo surface of degree 5;
- (iii) $\text{PSL}_2(7)$ acting on some special del Pezzo surface of degree 2 which can be realized as a double cover of \mathbb{P}^2 branched in the Klein quartic curve;
- (iv) \mathfrak{A}_5 acting on $\mathbb{P}^1 \times \mathbb{P}^1$ through the first factor.

3.2 Remark. It is known (see [DI09, §8] and [Che14, §B]) that the above listed actions are not conjugate in $\text{Cr}_2(\mathbb{C})$.

Proof. Applying the standard arguments (see e.g. [DI09, §3]) we may assume that G faithfully acts on a smooth projective rational surface X which is either a del Pezzo surface or an equivariant conic bundle. Moreover, one has $\text{rk Pic}(X)^G = 1$ (resp. $\text{rk Pic}(X)^G = 1$) in the del Pezzo (resp. conic bundle) case.

First, consider the case where X has an equivariant conic bundle structure $\pi : X \rightarrow B \simeq \mathbb{P}^1$. Since the group G is simple, it acts faithfully either on the base B or on the general fiber. Hence G is embeddable to $\text{PGL}_2(\mathbb{C})$. This is possible only if $G \simeq \mathfrak{A}_5$. We claim that π is a \mathbb{P}^1 -bundle. Assume that π has a degenerate fiber F . Then its components $F', F'' \subset F$ must be switched by an element $\alpha \in G$ of order 2. The intersection point $P = F' \cap F''$ is fixed by α and the action of α on the tangent space $T_{P,X}$ is diagonalizable. Since the tangent directions to F' and F'' are interchanged, the action of α on $T_{P,X}$ has the form $\text{diag}(1, -1)$. This means that α has a curve C of fixed points passing through P . Clearly, C dominates B and so the action of α on B is trivial. All elements of order 2 in $G \simeq \mathfrak{A}_5$ are conjugate and generate G . Hence G acts on B trivially. But then the point P must be fixed by G and G has a faithful two-dimensional representation on $T_{P,X}$, a contradiction.

Thus, π is a \mathbb{P}^1 -bundle and X is a Hirzebruch surface \mathbb{F}_n . Let F_1 be a fiber and let F_1, \dots, F_r be its orbit. By making elementary transformations

in F_1, \dots, F_r one obtains an equivariant birational maps $\mathbb{F}_n \dashrightarrow \mathbb{F}_{n+r}$ and (only if $n \geq r$) $\mathbb{F}_n \dashrightarrow \mathbb{F}_{n-r}$. There are such orbits for $r = 12, 20, 30$, and 60 . Using this trick we may replace \mathbb{F}_n with $\mathbb{F}_{n'}$, where $n' = 0$ or 1 . If $n' = 1$, then contracting the negative section we obtain the action on \mathbb{P}^2 with a fixed point. This is impossible for $G \simeq \mathfrak{A}_5$. Hence, we may assume that $X \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Additional elementary transformations allow to trivialize the action on the second factor. This is the case (iv). See [Che14, §B] for details.

From now on we assume that X is a del Pezzo surface* with $\text{rk Pic}(X)^G = 1$. We consider the possibilities according to the degree $d = K_X^2$.

Case $d = 1$. This case cannot occur, as $|-K_X|$ has one base point P , and G has to act on $T_{P,X}$ effectively. Hence $G \subset \text{GL}(T_{P,X})$. However, there are no simple finite subgroups in $\text{GL}_2(\mathbb{C})$, a contradiction.

Case $d = 2$. Then the anticanonical map $X \rightarrow \mathbb{P}^2$ is a double cover whose branch divisor $B \subset \mathbb{P}^2$ is a smooth quartic. The action of G in X descends to \mathbb{P}^2 so that B is G -stable. Therefore, $G \subset \text{Aut}(B)$. According to the Hurwitz bound $|G| \leq 168$. Moreover, $\text{Aut}(B)$ contains no elements of order 5. Then the only possibility is $G \simeq \text{PSL}_2(7)$ and $B = \{x_1^3x_2 + x_2^3x_3 + x_3^3x_1 = 0\}$. We get the case (iii).

Case $d = 3$. Then X is a cubic surface in \mathbb{P}^3 . The action of G on the lattice $\Lambda := K_X^\perp \subset \text{Pic}(X)$ is faithful. Hence our group G has a representation on the vector space $\Lambda/2\Lambda = (\mathbf{F}_2)^6$ over the field \mathbf{F}_2 . The intersection form induces an even quadratic form on Λ and, therefore, it induces a quadratic form

$$q(x) := \frac{1}{2}(x, x) \pmod{2}.$$

Take a standard basis $\mathbf{h}, \mathbf{e}_1, \dots, \mathbf{e}_6$ in $\text{Pic}(X)$ with $(\mathbf{h}, \mathbf{h}) = 1$, $(\mathbf{h}, \mathbf{e}_i) = 0$, $(\mathbf{e}_i, \mathbf{e}_j) = -\delta_j^i$. Then using the basis $\mathbf{e}_1 - \mathbf{e}_2$, $\mathbf{e}_2 - \mathbf{e}_3$, $\mathbf{e}_4 - \mathbf{e}_5$, $\mathbf{e}_5 - \mathbf{e}_6$, $\mathbf{h} - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3$, $\mathbf{h} - \mathbf{e}_4 - \mathbf{e}_5 - \mathbf{e}_6$ of Λ we can write $q(x)$ in the following form:

$$q(x) = x_1^2 + x_2^2 + x_1x_2 + x_3^2 + x_4^2 + x_3x_4 + x_5^2 + x_6^2 + x_5x_6.$$

Then it is easy to see that the Arf invariant of $q(x)$ equals 1. The group preserves the intersection form and the quadratic form $q(x)$. Therefore, there is a natural embedding $G \hookrightarrow \text{O}_6(\mathbf{F}_2)^-$. Since G is simple, $G \subset [\text{O}_6(\mathbf{F}_2)^-, \text{O}_6(\mathbf{F}_2)^-]$. It is known that $[\text{O}_6(\mathbf{F}_2)^-, \text{O}_6(\mathbf{F}_2)^-] \simeq \text{P}\mathfrak{Sp}_4(3)$ (see e.g. [CCN⁺85]). Moreover, G is isomorphic to one of the following groups: $\text{P}\mathfrak{Sp}_4(3)$, \mathfrak{A}_6 or \mathfrak{A}_5 . On the other hand, G faithfully acts on $H^0(X, -K_X) \simeq$

*More generally, actions of simple groups on del Pezzo surfaces with log terminal singularities were studied in [Bel15].

\mathbb{C}^4 . Then the only possibility is $G \simeq \mathfrak{A}_5$. The defining equation $\psi(z) = 0$ of $X \subset \mathbb{P}^3$ is a cubic invariant on $H^0(X, -K_X)$. Hence the representation on $H^0(X, -K_X)$ is the standard irreducible representation of \mathfrak{A}_5 . Then in a suitable basis we have $\psi = z_1^3 + \cdots + z_4^3 - (z_1 + \cdots + z_4)^3$. The fixed point locus of an element $\alpha \in G$ of order two is a union of a line and three isolated points. By the topological Lefschetz fixed point formula the action of α on Λ is diagonalizable as follows: $\alpha = \text{diag}(1, 1, 1, 1, -1, -1)$. This implies that the representation of G on Λ is the sum of the irreducible four-dimensional representation and the trivial one. This contradicts the minimality assumption $\text{rk Pic}(X)^G = 1$.

Case $d = 4$. Then $X = X_{2,2} = Q' \cap Q'' \subset \mathbb{P}^4$ is an intersection of two quadrics. The group G acts on the pencil of quadrics $\langle Q', Q'' \rangle \simeq \mathbb{P}^1$ leaving invariant the subset of five singular elements. It is easy to see that in this case the action on $\langle Q', Q'' \rangle$ must be trivial. Hence, G fixes vertices P_1, \dots, P_5 of five G -stable quadratic cones $Q_i \in \langle Q', Q'' \rangle$. Since these points P_1, \dots, P_5 generate \mathbb{P}^5 , G must be abelian, a contradiction.

Case $d = 5$. A del Pezzo surface of degree 5 is unique up to isomorphism. Consider the (faithful) action of G on the space $\text{Pic}(X) \otimes \mathbb{C}$ and on the orthogonal complement $K_X^\perp \subset \text{Pic}(X) \otimes \mathbb{C}$. The intersection form induces a non-degenerate quadratic form on K_X^\perp . Hence G faithfully acts on a two-dimensional quadric in \mathbb{P}^3 . Then the only possibility is $G \simeq \mathfrak{A}_5$. One can see that $\text{Aut}(X)$ is isomorphic to the symmetric group \mathfrak{S}_5 and so a del Pezzo surface of degree 5 admits an action of \mathfrak{A}_5 . We get the case (ii).

Case $6 \leq d \leq 8$. Then the action of G on $\text{Pic}(X) \simeq \mathbb{Z}^{10-d}$ must be trivial. This contradicts the minimality assumption $\text{rk Pic}(X)^G = 1$.

Case $d = 9$. Then $X = \mathbb{P}^2$. So $G \subset \text{PGL}_3(\mathbb{C})$, and by the classification of finite subgroups in $\text{PGL}_2(\mathbb{C})$ we get the case (i). Theorem 3.1 is proved. \square

3.3 Theorem ([DI09], [Tsy13]). *Let $G \subset \text{Cr}_2(\mathbb{C})$ be a finite quasi-simple non-simple subgroup. Then $G \simeq 2.\mathfrak{A}_5$.*

Proof. As in the proof of Theorem 3.1 we may assume that G faithfully acts on a smooth projective rational surface X which is either a del Pezzo surface with $\text{rk Pic}(X)^G = 1$ or an equivariant conic bundle with $\text{rk Pic}(X)^G = 2$. Let $\bar{G} := G/\text{z}(G)$. Then \bar{G} is a simple group acting on a rational surface $X/\text{z}(G)$. Hence \bar{G} is embeddable to $\text{Cr}_2(\mathbb{C})$ and by Theorem 3.1 we have $\bar{G} \simeq \mathfrak{A}_5, \mathfrak{A}_6$, or $\text{PSL}_2(7)$. Therefore, as in the proof of Proposition 2.1 we have one of the following possibilities: $G \simeq 2.\mathfrak{A}_5, \text{SL}_2(7)$, or $n.\mathfrak{A}_6$ for $n = 2, 3$, or 6 . If X has an equivariant conic bundle structure $\pi : X \rightarrow B \simeq \mathbb{P}^1$, then G non-trivially acts either on the base B or on the general fiber. This is possible only if $G \simeq 2.\mathfrak{A}_5$.

Assume that X is a del Pezzo surface with $\text{rk Pic}(X)^G = 1$. Let $Z \subset \text{z}(G)$ be a cyclic subgroup of prime order p and let $\pi : X \rightarrow Y := X/Z$ be the quotient. The surface Y is rational and $\bar{G} := G/Z$ faithfully acts on Y .

First, consider the case where Z has only isolated fixed points. If $p = 2$, then by the holomorphic Lefschetz formula the number of fixed points equals 4. These points cannot be permuted by G , so they are fixed by G . Similarly, in the case $p = 3$ denote by n_0 (resp., n_1, n_2) the number of fixed points with action of type $\frac{1}{3}(1, -1)$ (resp., $\frac{1}{3}(1, 1), \frac{1}{3}(-1, -1)$). Then again by the holomorphic Lefschetz formula $n_1 = n_2, n_0 + n_1 = 3$. Hence there are at most three points of each type and so, as above, all these points are fixed by G . Since the groups $\text{SL}_2(7)$ and $n.\mathfrak{A}_6$ cannot act faithfully on the tangent space to a fixed point, the only possibility is $G \simeq 2.\mathfrak{A}_5$.

Now consider the case where the fixed point locus X^Z of Z is one-dimensional. Let C be the union of all the curves in X^Z . Note that C is smooth because it is a part of the fixed point locus. Clearly, C is G -invariant. Then the class of C must be proportional to $-K_X$ in $\text{Pic}(X)$. Hence C is ample and connected. Since C is smooth, it is irreducible. The group G/Z non-trivially acts on C . Hence C cannot be an elliptic curve. Moreover, if C is rational, then $G/Z \simeq \mathfrak{A}_5$ and we are done. Thus we can write $C \sim -aK_X$ with $a > 1$. If $-K_X$ is very ample, then C is contained in a hyperplane section and $a = 1$, a contradiction. Thus it remains to consider only two possibilities: $K_X^2 = 1$ and 2. If $K_X^2 = 1$, then the anticanonical linear system has a unique base point, say O . Since, the representation of G in the tangent space $T_{O,X}$ is faithful, the only possibility is $G \simeq 2.\mathfrak{A}_5$. Finally assume that $K_X^2 = 2$. Then the anticanonical map is a double cover $\Phi_{|-K_X|} : X \rightarrow \mathbb{P}^2$. The action of Z of \mathbb{P}^2 must be trivial (otherwise $a = 1$). Hence $p = 2$, Z is generated by the Geiser involution γ , and C is the ramification curve of $\Phi_{|-K_X|}$. On the other hand, there exists the following homomorphism

$$\lambda : \text{Aut}(X) \hookrightarrow \text{GL}(\text{Pic}(X)) = \text{GL}_8(\mathbb{Z}) \xrightarrow{\det} \{\pm 1\},$$

where $\lambda(\gamma) = -1$. Since our group G is perfect, $\gamma \notin G$, a contradiction. \square

3.4 Remark. In the same manner one can describe the actions of $G = 2.\mathfrak{A}_5$ on rational surfaces, i.e. the embeddings $2.\mathfrak{A}_5 \hookrightarrow \text{Cr}_2(\mathbb{C})$, see [Tsy13] for details.

Acknowledgements. I would like to thank the referee for careful reading and numerous helpful comments and suggestions.

References

- [AW97] Dan Abramovich and Jianhua Wang. Equivariant resolution of singularities in characteristic 0. *Math. Res. Lett.*, 4(2-3):427–433, 1997.
- [Bea14] Arnaud Beauville. Finite simple groups of small essential dimension. In *Trends in contemporary mathematics.. Selected talks based on the presentations at the INdAM day, June 18, 2014*, pages 221–228. Cham: Springer, 2014.
- [Bel15] Grigory Belousov. Log del Pezzo surfaces with simple automorphism groups. *Proc. Edinb. Math. Soc., II. Ser.*, 58(1):33–52, 2015.
- [BR97] J. Buhler and Z. Reichstein. On the essential dimension of a finite group. *Compos. Math.*, 106(2):159–179, 1997.
- [CCN⁺85] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, and R.A. Wilson. *Atlas of finite groups. Maximal subgroups and ordinary characters for simple groups. With comput. assist. from J. G. Thackray*. Oxford: Clarendon Press. XXXIII, 252 p., 1985.
- [Che14] I. A. Cheltsov. Two local inequalities. *Izv. Math.*, 78(2):375–426, 2014.
- [DI09] Igor V. Dolgachev and Vasily A. Iskovskikh. Finite subgroups of the plane Cremona group. In *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. I*, volume 269 of *Progr. Math.*, pages 443–548. Birkhäuser Boston Inc., Boston, MA, 2009.
- [Dun10] Alexander Duncan. Essential dimensions of A_7 and S_7 . *Math. Res. Lett.*, 17(2):263–266, 2010.
- [Dun13] Alexander Duncan. Finite groups of essential dimension 2. *Comment. Math. Helv.*, 88(3):555–585, 2013.
- [Kar93] Gregory Karpilovsky. *Group representations. Vol. 2*, volume 177 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1993.
- [KM08] Nikita Karpenko and Alexander Merkurjev. Essential dimension of finite p -groups. *Invent. math.*, 172:491–508, 2008.

- [KS00] János Kollár and Endre Szabó. Fixed points of group actions and rational maps. *Canad. J. Math.*, 52(5):1054–1056, 2000. Appendix to “Essential dimensions of algebraic groups and a resolution theorem for G -varieties” by Z. Reichstein and B. Youssin.
- [Pro12] Yu. Prokhorov. Simple finite subgroups of the Cremona group of rank 3. *J. Algebraic Geom.*, 21(3):563–600, 2012.
- [PS16a] Yu. Prokhorov and C. Shramov. Finite groups of birational self-maps of threefolds. *ArXiv e-print*, 1611.00789, 2016.
- [PS16b] Yu. Prokhorov and C. Shramov. Jordan constant for Cremona group of rank 3. *ArXiv e-print*, 1608.00709, 2016.
- [PS16c] Yu. Prokhorov and C. Shramov. p -subgroups in the space Cremona group. *ArXiv e-print*, 1610.02990, 2016.
- [Ser10] Jean-Pierre Serre. Le groupe de Cremona et ses sous-groupes finis. In *Séminaire Bourbaki. Volume 2008/2009. Exposés 997–1011*, pages 75–100, ex. Paris: Société Mathématique de France (SMF), 2010.
- [Tsy13] Vladimir Igorevich Tsygankov. The conjugacy classes of finite nonsolvable subgroups in the plane Cremona group. *Adv. Geom.*, 13(2):323–347, 2013.